

Intern. J. Computer Math., Vol. 23, pp. 265–282
Photocopying permitted by license only
Reprints available directly from the publisher
© 1988 Gordon and Breach, Science Publishers, Inc.
Printed in Great Britain

Several New Methods for Solving Equations

BENY NETA

*Naval Postgraduate School, Department of Mathematics, Code N3Nd,
Monterey, CA 93943, USA*

(Received November 1986; in final form June 1987)

Several new methods for solving one nonlinear equation are developed. Most of the methods are of order three and they require the knowledge of f , f' and f'' . The methods will be compared to others in the literature. An extensive bibliography is given.

KEY WORDS: Nonlinear equations, order of convergence, iteration, computational efficiency, efficiency index.

C.R. CATEGORIES: 5.1, 5.15.

1. INTRODUCTION

Many iterative procedures were developed to obtain a simple zero ξ of a nonlinear function $f(x)$. There are also methods for obtaining multiple zeros. The algorithms can be classified as bracketing techniques, fixed point methods and hybrid ones. The bracketing methods include the well-known bisection, Regula Falsi and modified Regula Falsi. Other algorithms of this class are: method F (King [66]), modified method F (Popovski [108]), Illinois algorithm (Snyder [116]), Pegasus (Dowell and Jarratt [24]), and improved Pegasus method (King [64]), algorithms A, M and R (Dekker [21]; Bus and Dekker [11]), algorithm B (Brent [9]), algorithm C

(Anderson and Bjorck [5]), Cox method (Cox [18]), and Stone's method (Stone [120]). In all these methods one assumes that an interval $[a, b]$ is given on which $f(x)$ changes its sign. The methods successively produce smaller and smaller intervals containing the zero. Thus one guarantees the convergence of the iterative process. On the other hand such methods *cannot* find zeros of even multiplicity. In order to overcome such difficulty one can use fixed point type method. The list of such methods is long and includes among others Cauchy, Chebyshev, Euler, Halley, Hansen and Patrick, Jarratt, King, Laguerre, Muller, Murakami, Neta, Newton, Ostrowski, Popovski, Steffensen, secant, Traub, Wegstein and Werner.

The last class uses a combination of two methods, one from each class to guarantee the convergence of the iterative process, see for example Nesdore [79] and Popovski [86–108].

In this article, we develop several new fixed-point type methods based on the idea of Popovski [107]. All such methods will require the evaluation of f , f' and f'' at each step. These methods are all of order three and thus the informational efficiency (see e.g., Ostrowski) is 1. The efficiency index is 1.442. A fourth-order method based on Nourein's algorithm [83] will be given. A special case of a fifth-order method developed by Murakami [77] will also be discussed. These methods will be compared numerically. Tables comparing the informational efficiency and the efficiency index of all methods (known to the author) will be given.

2. THIRD-ORDER METHODS

In 1982 Popovski has suggested the construction of third-order methods by what he called the method of replacement. Let

$$h = x_{n+1} - x_n, \quad f_n = f(x_n), \quad u_n = f_n/f'_n, \quad A_i = \frac{f^{(i)}(x_n)}{i! f''(x_n)},$$

then

$$0 = u + h + A_2 h^2. \quad (1)$$

This equation can be solved for h directly, which yields Cauchy's

method [13]. If (1) is written as

$$0 = u + h + A_2(h) \cdot \{h\} \quad (2)$$

one can replace $\{h\}$ and $\{h\}$ by various iteration functions and then solve for h . This is the method of replacement. Popovski obtained the following 9 algorithms this way.

$$1. \quad h = -\frac{u(uA_2 - 1)}{2uA_2 - 1}, \quad (3)$$

$$2. \quad h = \frac{u}{(uA_2 + 1)uA_2 - 1}, \quad (4)$$

$$3. \quad h = \frac{u(2uA_2 - 1)}{(uA_2 - 3)uA_2 + 1}, \quad (5)$$

$$4. \quad h = -\frac{u[(uA_2 - 3)uA_2 + 1]}{3(uA_2 - 4)uA_2 + 1}, \quad (6)$$

$$5. \quad h = -\frac{u[(uA_2 + 1)uA_2 - 1]}{2uA_2 - 1}, \quad (7)$$

$$6. \quad h = u \left(\frac{uA_2}{(uA_2 + 1)uA_2 - 1} - 1 \right), \quad (8)$$

$$7. \quad h = \frac{u[(uA_2 + 2)uA_2 - 1]}{(uA_2 - 3)uA_2 + 1}, \quad (9)$$

$$8. \quad h = -u \frac{uA_2}{(uA_2 - 1)^2 + 1}, \quad (10)$$

$$9. \quad h = u \frac{(uA_2)^2 + 1}{uA_2 - 1}. \quad (11)$$

These methods based on combining $(h) = h, -u$ or one of the following methods

$$h = \frac{u}{uA_2 - 1} \quad (\text{Halley}), \quad (12)$$

$$h = -u(uA_2 + 1) \quad (\text{Euler}), \quad (13)$$

$$h = -u[(2uA_2 + 1)uA_2 + 1], \quad (14)$$

and (3)–(5).

In a similar fashion, one can obtain the following 21 new methods.

$$h = -\frac{u}{1-uA_2(1+uA_2(1+2uA_2))}, \quad (h), \quad (14) \quad (15)$$

$$h = -\frac{u}{1+uA_2/[(1+uA_2)uA_2-1]}, \quad (h), \quad (4) \quad (16)$$

$$h = -u - u^2 A_2 [(2uA_2 + 1)uA_2 + 1], \quad (-u), \quad (12) \quad (17)$$

$$h = -u - u^2 A_2 (uA_2 + 1), \quad (-u), \quad (13) \quad (18)$$

$$h = -u - u^2 A_2 [(2uA_2 + 1)uA_2 + 1], \quad (-u), \quad (14) \quad (19)$$

$$h = -u + u^2 A_2 \frac{(2uA_2 + 1)uA_2 + 1}{uA_2 - 1}, \quad (12), \quad (14) \quad (20)$$

$$h = -u - \frac{u^2 A_2}{(uA_2 - 1)[(uA_2 + 1)uA_2 - 1]}, \quad (12), \quad (4) \quad (21)$$

$$h = -u - \frac{u^2 A_2}{uA_2 - 1} \frac{2uA_2 - 1}{(uA_2 - 3)uA_2 + 1}, \quad (12), \quad (5) \quad (22)$$

$$h = -u - u^2 A_2 (uA_2 + 1)^2, \quad (13), \quad (13) \quad (23)$$

$$h = -u - A_2 u^2 (uA_2 + 1) [(2uA_2 + 1)uA_2 + 1], \quad (13), \quad (14) \quad (24)$$

$$h = -u - A_2 \frac{u^2 (uA_2 + 1)(uA_2 - 1)}{2uA_2 - 1}, \quad (13), \quad (3) \quad (25)$$

$$h = -u + A_2 \frac{u^2 (uA_2 + 1)(2uA_2 - 1)}{(uA_2 - 3)uA_2 + 1}, \quad (13), \quad (5) \quad (26)$$

$$h = -u - A_2 u^2 [(2uA_2 + 1)uA_2 + 1]^2, \quad (14), \quad (14) \quad (27)$$

$$h = -u - A_2 \frac{u^2 (uA_2 - 1) [(2uA_2 + 1)uA_2 + 1]}{2uA_2 - 1}, \quad (14), \quad (3) \quad (28)$$

$$h = -u + A_2 \frac{u^2[(2uA_2+1)uA_2+1]}{(uA_2+1)uA_2-1}, \quad (14), \quad (4) \quad (29)$$

$$h = -u + A_2 \frac{u^2(2uA_2-1)[(2uA_2+1)uA_2+1]}{(uA_2-3)uA_2+1}, \quad (14), \quad (5) \quad (30)$$

$$h = -u - A_2 \left[\frac{u(uA_2-1)}{(2uA_2-1)} \right]^2, \quad (3), \quad (3) \quad (31)$$

$$h = -u + A_2 \frac{u^2(uA_2-1)}{(2uA_2-1)[(uA_2+1)uA_2-1]}, \quad (3), \quad (4) \quad (32)$$

$$h = -u - A_2 \left[\frac{u}{(uA_2+1)uA_2-1} \right]^2, \quad (4), \quad (4) \quad (33)$$

$$h = -u - A_2 \frac{u^2(2uA_2-1)}{[(uA_2+1)uA_2-1][(uA_2-3)uA_2+1]}, \quad (4), \quad (5) \quad (34)$$

$$h = -u - A_2 \left[\frac{u(2uA_2-1)}{(uA_2-3)uA_2+1} \right]^2. \quad (5), \quad (5) \quad (35)$$

Remark The two quantities in parentheses to the left of the equation number indicate how the new method was developed.

Since Popovski [107] recommended the use of method (3) we compare that method with the newly developed ones in Section 5.

3. FOURTH-ORDER METHOD

This method is based on Nourein's algorithm [83]. The method of neglecting discussed by Popovski [107] is used here. Setting $f^{(4)}(x_i)=0$, in Nourein's method one obtains

$$h = \frac{(2uA_2-1-6u^2A_3)u}{(uA_2-3)uA_2+1+6u^2A_3}. \quad (36)$$

The method is of order four and requires four new pieces of

information. Therefore the informational efficiency is 1 as for the previous methods. On the other hand the efficiency index is only 1.414.

4. FIFTH-ORDER METHOD

Murakami [77] has developed the following family of methods of order five,

$$h = -a_1 u_n - a_2 w_2(x_n) - a_3 w_3(x_n) - \psi(x_n), \quad (37)$$

where

$$w_2(x_n) = \frac{f_n}{f'(x_n - u_n)}, \quad (38)$$

$$w_3(x_n) = \frac{f_n}{f'(x_n + \beta u_n + \gamma w_2(x_n))}, \quad (39)$$

$$\psi(x_n) = \frac{f_n}{b_1 f'_n + b_2 f''_n(x_n - u_n)}. \quad (40)$$

The asymptotic error constant is

$$C = -\left[\frac{\frac{32}{3}\gamma^2 + \frac{8}{3}\gamma - \frac{2}{3}}{6(4\gamma + 1)} + \frac{1}{6(4\gamma + 1)} \right] A_2^4 + (8\gamma^2 + 4\gamma) A_2^2 A_3 \\ + \frac{128}{3}\gamma A_2 A_4 - \frac{3}{8}A_3^2 + \frac{1}{24}A_5. \quad (41)$$

Murakami has suggested that $\gamma = 0$ or $\gamma = -\frac{1}{2}$. These two choices annihilate one term of the asymptotic error constant. Another possibility is $\gamma = 17795/131072$ which annihilates the first term. This choice leads to the values

$$\begin{aligned} \beta &= -\frac{1}{2} - \gamma, \quad a_1 = 0.3879870, \quad a_2 = -1.420700, \\ a_3 &= \frac{2}{3}, \quad b_1 = -0.1186015, \quad b_2 = 0.8506410, \end{aligned} \quad (42)$$

and the asymptotic error constant is then

$$C = 0.6905251A_2^2A_3 + 5.792695A_2A_4 - \frac{3}{8}A_3^2 + \frac{1}{24}A_5. \quad (43)$$

5. NUMERICAL EXPERIMENTS

We have used the following six examples to compare the performance of the methods (3), (15)–(37).

$$f_1(x) = \sin x - \frac{1}{2}x, \quad x_0 = 2$$

$$f_2(x) = x^5 + x - 10\,000, \quad x_0 = 4$$

$$f_3(x) = x^{1/2} - \frac{1}{x} - 3, \quad x_0 = 1$$

$$f_4(x) = e^x + x - 20, \quad x_0 = 0$$

$$f_5(x) = \ln x + x^{1/2} - 5, \quad x_0 = 1$$

$$f_6(x) = x^3 - x^2 - 1, \quad x_0 = 0.5.$$

The first example is simple and it's taken from Gerald and Wheatley [35]. All methods performed very well in this case. The other examples are taken from Popovski [86] and thus we added the method of that article to our comparison. In our notations, the method is

$$h = -\frac{e^{2uA_2} - 1}{2A_2}. \quad (44)$$

The following table (Table 1) gives the number of iterations required to obtain the zero with tolerance of 10^{-14} . All computations were performed in double precision on an IBM 3033. The letter D stands for divergence (within 30 iterations) and * denotes computational difficulties (overflow, etc.). Note that the method recommended in Popovski [107] didn't perform as well as the new

Table 1

Method	Function					
	1	2	3	4	5	6
(3)	3	6	D	*	D	6
(15)	3	10	D	8	D	D
(16)	3	D	*	D	*	13
(17)	3	29	4	14	4	16
(18)	3	12	4	26	4	17
(19)	3	29	4	14	4	16
(20)	3	16	*	10	*	11
(21)	3	6	5	6	5	12
(22)	3	7	*	D	*	7
(23)	3	14	5	13	5	14
(24)	3	16	7	*	6	19
(25)	3	8	5	*	5	23
(26)	3	D	4	D	4	16
(27)	3	20	*	11	5	19
(28)	3	D	4	*	4	11
(29)	3	9	*	25	*	13
(30)	3	D	5	D	5	8
(31)	3	11	6	5	5	5
(32)	3	15	5	*	5	11
(33)	3	6	7	9	4	7
(34)	3	D	D	D	D	D
(35)	3	6	4	7	4	6
(36)	3	6	5	8	5	15
(37)	2	5	4	4	3	7
(44)	3	6	4	5	4	D

methods developed here. The method taken from Popovski [86] did not converge for one of the examples. The following new methods converge for all the examples (17)–(19), (21), (23), (31), (33), (35)–(37). Counting the total number of iterations one can say that methods (21), (31), (33) (35) are the best third-order ones of those considered. Next comes (23), and then (17)–(19). Certainly method (37) which is fifth-order performed better than the others. Surprisingly though the fourth-order method (36) didn't perform better than the third-order methods.

Remark More experiments were performed with functions suggested by Nerinckx and Haegemans [78] and the results were consistently better than those reported there.

6. EFFICIENCY COMPARISON

In this section we collected information concerning the order, informational usage, informational efficiency and efficiency index (for definitions, see e.g. Neta [81]) of all methods known to the author. The first table (Table 2) consists of the information for bracketing methods and the others will give fixed point methods (Tables 3-7). For the fixed point methods we gave separate tables for derivative free methods, for methods using f' at only one point, for those using f' at more than one point, for those using f'' and f''' and those using derivatives of order 3 or higher.

Table 2

	P	d	E	I	Total no.	f'
Bisection	1	1	1	1	$N = \log_2 \frac{ b-a }{\epsilon}$	
Regula Falsi	1	1	1	1		
Modified R.F.	1.618	1	1.618	1.618		
Algorithm A	1.618	1	1.618	1.618	2^N	
Algorithm B	1.618	1	1.618	1.618	$(N+1)^2 - 2$	
Algorithm M	1.618	1	1.618	1.618	$4N$	
Algorithm R	1.839	1	1.839	1.839	$5N$	
Method F	1.839	1	1.839	1.839		
Modified						
Method F	1.839	1	1.839	1.839		
Illinois	3	3	1	1.442		
Pegasus	7.275	4	1.818	1.642		
Improved	3	2	1.5	1.732		
Pegasus	5	3	1.667	1.710		
Algorithm C	5	3	1.667	1.710		
	8	4	2	1.682		
Cox	2	2	1	1.414	1	
Stone	3	2	1.5	1.732	1	

In the last two algorithms we assumed equal complexity in evaluating f, f' . The efficiency may be higher in other cases.

Table 3

	<i>P</i>	<i>d</i>	<i>E</i>	<i>I</i>
Fixed point (Picard)	1	1	1	1
Wegstein	1.618	1	1.618	1.618
Secant	1.618	1	1.618	1.618
Popovski [98]	1.839	1	1.839	1.839
Muller	1.839	1	1.839	1.839
Jarratt and Nudds	1.839	1	1.839	1.839
Popovski [100]	1.839	1	1.839	1.839
Traub (3 methods)	1.839	1	1.839	1.839
Steffensen	2	2	1	1.414
Chambers	2	2	1	1.414
Chambers	2.732	2	1.366	1.653

Only *f* values are required.

Table 4

	<i>P</i>	<i>d</i>	<i>E</i>	<i>I</i>
Newton	2	2	1	1.414
Dordjevic	2	2	1	1.414
Ostrowski	2.414	2	1.207	1.554
Popovski [99]	2.414	2	1.207	1.554
Popovski [105] 3 methods	2.414	2	1.207	1.554
Werner [131]	2.414	2	1.207	1.554
Chambers [15]	2.414	2	1.207	1.554
Jarratt [53]	2.732	2	1.366	1.653
Jain (implicit)	3	-	-	-
Werner [134]	3	3	1	1.442
King [65]	4	3	1.333	1.587
Popovski [105] 10 methods	4.562	3	1.520	1.66
Murakami	5	4	1.25	1.495
Neta (this article)	5	4	1.25	1.495
Neta [80]	6	4	1.5	1.565
Popovski [104]	7	4	1.75	1.626
Neta [82]	10.815	4	2.704	1.813
Neta [81]	14	5	2.8	1.695
Neta [81]	16	5	3.2	1.741
Werner [133]	$2k$	$k+2$	-	-
Werner [133]	$k + \sqrt{k^2 + 1}$	$k+2$	-	-
Werner [133]	$2k+1$	$k+2$	-	-
Werner [133]	$2k+2$	$k+2$	-	-

f' is required at one point only.

Table 5

	<i>P</i>	<i>d</i>	<i>E</i>	<i>I</i>	<i>f' at</i>
Jarratt [52]	3	3	1	1.442	2
Jarratt [55]	4	3	1.333	1.587	2
Jarratt [51] 4 methods	4	3	1.333	1.587	2
Jarratt [52] 2 methods	4	4	1	1.414	3
Jain (semi-explicit)	4	-	-	-	2
Jarratt [52]	5	4	1.25	1.495	3
Jain (implicit)	5	-	-	-	2
King [62]	5	4	1.25	1.495	2
Popovski [101]	7.464	-	4	1.866	1.653
Werner	$\frac{m}{2} + \sqrt{\frac{m^2}{4} + 1}$	m	-	-	$m - 1$

Methods require *f* and *f'* at more than one point.

Table 6

	<i>P</i>	<i>d</i>	<i>E</i>	<i>I</i>
Hansen and Patrick	3	3	1	1.442
Popovski [93]	3	3	1	1.442
Halley	3	3	1	1.442
Laguerre	3	3	1	1.442
Chebyshev	3	3	1	1.442
Cauchy	3	3	1	1.442
Euler	3	3	1	1.442
Ostrowski	3	3	1	1.442
Popovski [89]	3	3	1	1.442
Milovanovic <i>et al.</i>	3	3	1	1.442
Popovski [90]	3	3	1	1.442
Popovski [94]	3	3	1	1.442
Neta (this article) 21 methods	3	3	1	1.442
Popovski [107] 9 methods	3	3	1	1.442
Werner [132]	4	3	1.333	1.587
Werner [133] 4 methods	4	3	1.333	1.587
Popovski [106]	6	4	1.5	1.565

Methods require *f'* and *f''*.

Table 7

	<i>P</i>	<i>d</i>	<i>E</i>	<i>I</i>	<i>f'</i>	<i>f''</i>	<i>f'''</i>	<i>f⁽⁴⁾...f^(k)</i>
Kiss/Lika	4	4	1	1.414	1	1	1	
Neta (this article)	4	4	1	1.414	1	1	1	
Nourein	5	5	1	1.380	1	1	1	1
Werner	<i>k</i> +2	<i>k</i> +1		—	1	1	1	1 ... 1
Varyukhin and Kasyanyuk	<i>k</i> +2	<i>k</i> +2	—	—	1	1	1	1 ... 1

Methods requiring derivatives of *f* of order three or higher.

Acknowledgement

The author would like to thank the NPS Foundation Research Program for its support of this research.

References

- [1] A. C. Aitken, On Bernoulli's numerical solution of algebraic equations, *Proc. Roy. Soc. Edinburgh* **46** (1926), 289–305.
- [2] A. C. Aitken, Further numerical studies in algebraic equations, *Proc. Roy. Soc. Edinburgh* **51** (1931), 80–90.
- [3] M. Altman, A generalization of Newton's method, *Bull. Acad. Polon. Sci., Cl III* **3** (1955), 189–193.
- [4] M. Altman, Iterative methods of higher order, *Bull. Acad. Sci. Polon. Sci. Ser. Math. Astr. Phys.* **9** (1961), 63–68.
- [5] N. Anderson and A. Bjorck, A new high order method of Regula Falsi type for computing a root of an equation, *BIT* **13** (1973), 253–264.
- [6] E. H. Bateman, Halley's methods of solving equations, *Amer. Math. Monthly* **45** (1938), 11–17.
- [7] E. H. Bateman, The solution of algebraic and transcendental equations by iteration, *Math. Gaz.* **37** (1953), 96–101.
- [8] E. Bodewig, Sur la method Laguerre pour l'approximation des racines de certaines équations algébriques et sur la critique d'Hermite, *Indag. Math.* **8** (1946), 570–580.
- [9] R. P. Brent, An algorithm with guaranteed convergence for finding a zero of a function, *Computer J.* **14** (1971), 422–425.
- [10] R. P. Brent, S. Winograd and P. Wolfe, Optimal iterative processes for root-finding, *Numer. Math.* **20** (1973), 327–341.
- [11] J. C. P. Bus and T. J. Dekker, Two efficient algorithms with guaranteed convergence for finding a zero of a function, Mathematical Centre Report NW 13/74, Amsterdam, Sept. 1974, 27 pages, *Assoc. Comput. Mach. TOMS* **1** (1975), 330–345.

- [12] E. Carvallo, Méthode pratique pour la résolution numérique des équations algébriques ou transcendantes, Doctoral thesis, Paris, 1890.
- [13] A. Cauchy, Sur la résolution numérique des équations algébriques et transcendantes, *C.R. Acad. Sci. Paris* **11** (1840), 829–847.
- [14] A. Cauchy, Sur la détermination approximative des racines d'une équation algébrique ou transcendante, *Oeuvres Complètes Serie 2(4)*, 573–609, Gauthier Villars, Paris, 1899.
- [15] L. G. Chambers, A quadratic formula for finding the root of an equation, *Math. Comp.* **25** (1971), 305–307.
- [16] M. L. Chambers and P. Jarratt, Use of double sampling for selecting best population, *Biometrika* **51** (1964), 1 and 2, 49–64.
- [17] J. G. van der Corput, Sur l'approximation de Laguerre des racines d'une équation qui a toutes ses racines réelles, *Nederl. Akad. Wetensch. Proc.* **49** (1946), 922–929.
- [18] M. G. Cox, A bracketing technique for computing a zero of a function, *Computer J.* **13** (1970), 101–102.
- [19] M. G. Cox, A note on Chamber's method for finding a zero of a function, *Math. Comp.* **26** (1972), 749.
- [20] M. Davies and B. Dawson, On the global convergence of Halley's iteration formula, *Numer. Math.* **24** (1975), 133–135.
- [21] T. J. Dekker, Finding a zero by means of successive linear interpolation. In: *Constructive Aspects of the Fundamental Theorem of Algebra*, B. Dejon and P. Henrici (eds.), Wiley Interscience, London, 1969.
- [22] L. N. Dordjević, An iterative solution of algebraic equations with a parameter to accelerate convergence, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **449** (1973), 179–182.
- [23] M. Dowell and P. Jarratt, A modified Regula Falsi method of computing the root of an equation, *BIT* **11** (1971), 168–174.
- [24] M. Dowell and P. Jarratt, The "Pegasus" method for computing the root of an equation, *BIT* **12** (1972), 503–508.
- [25] E. Durand, Solutions numériques des équations algébriques, *Masson et Cie* **2** (1960), Paris.
- [26] G. E. Forsythe, Note on high-order algorithms for solving transcendental equations, Unpublished report, Stanford University, 1957.
- [27] L. V. Foster, Generalizations of Laguerre's method: Higher order methods, *SIAM J. Numer. Anal.* **18** (1981), 1004–1018.
- [28] J. S. Frame, A variation of Newton's method, *Amer. Math. Monthly* **51** (1944), 36–38.
- [29] J. S. Frame, Remarks on a variation of Newton's method, *Amer. Math. Monthly* **52** (1945), 212–214.
- [30] E. Frank, On the calculations of the roots of equations, *J. Math. and Phys.* (now *Studies in Appl. Math.*) **34** (1955), 187–197.
- [31] W. L. Frank, Finding zeros of arbitrary functions, *J. Assoc. Comput. Mach.* **5** (1958), 154–160.
- [32] L. Galeone, Generalizzazione del metodo di Laguerre, *Calcolo* **14** (1977), 121–131.

- [33] L. Galeone, Un metodo iterativo non stazionario per la risoluzione di equazioni algebriche, *Calcolo* **15** (1978), 289–298.
- [34] A. N. Gleyzal, Solution of nonlinear equations, *Quart. Appl. Math.* **17** (1959), 95–96.
- [35] C. F. Gerald and P. O. Wheatley, *Applied Numerical Analysis*, Addison Wesley Pub. Co., Reading, MA, 1985.
- [36] H. Gorecki and A. B. Turowicz, Sur la resolution des equations algebriques par la methode de Euler, *Ann. Polon. Math.* **12** (1962), 185–190.
- [37] M. S. Gornstein, The numerical solution of equations (Russian), *Dokl. Akad. Nauk SSSR* **78** (1951), 193–196.
- [38] D. Grave, Algorithme du calcul des racines des equations algebriques, *J. Math. Inst. Ukrainian Academy of Sciences* **2** (1936), 3–20.
- [39] E. Halley, A new, exact and easy method of finding the roots of equations generally and that without any previous reduction, *Phil. Trans. Roy. Soc. London* **18** (1694), 136–148.
- [40] H. J. Hamilton, Roots of equations by functional iteration, *Duke Math. J.* **13** (1946), 113–121.
- [41] H. J. Hamilton, A type of variation on Newton's method, *Amer. Math. Monthly* **57** (1950), 517–522.
- [42] E. Hansen and M. Patrick, A family of root finding methods, *Numer. Math.* **27** (1977), 257–269.
- [43] P. Hartman, Newtonian approximations to a zero of a function, *Comm. Math. Helv.* **21** (1948), 321–326.
- [44] D. R. Hartree, Notes on iterative processes, *Proc. Cambridge Phil. Soc.* **45** (1949), 230–236.
- [45] S. Hitotumatu, A method of successive approximation based on the expansion of second order, *Math. Japon.* **7** (1972), 31–50.
- [46] U. Hochstrasser, Numerical methods for finding solutions of nonlinear equations. In: *A Survey of Numerical Analysis*, J. Todd (ed.), McGraw-Hill Book Company, Inc., New York, 1962, pp. 255–278.
- [47] D. J. Hofsmmer, Note on the computation of the zeros of functions satisfying a second order differential equation, *Math. Table & Other Aids Comp.* **12** (1958), 58–60.
- [48] A. S. Householder, Polynomial iterations to roots of algebraic equations, *Proc. Amer. Math. Soc.* **2** (1951), 718–719.
- [49] L. C. Hsu, A few useful modifications of Newton's approximation method of solving real equations, *Math. Student* **26** (1958), 145–153.
- [50] P. Jarratt and D. Nudds, The use of rational functions in the iterative solution of equations on a digital computer, *Computer J.* **8** (1965), 62–65.
- [51] P. Jarratt, Some fourth order multipoint methods for solving equations, *Math. Comp.* **20** (1966), 434–437.
- [52] P. Jarratt, Multipoint iterative methods for solving certain equations, *Computer J.* **8** (1966), 398–400.
- [53] P. Jarratt, A rational iteration function for solving equations, *Computer J.* **9** (1966), 304–307.
- [54] P. Jarratt, A note on the asymptotic error constant of a certain method for solving equations, *Computer J.* **9** (1967), 408–409.

- [55] P. Jarratt, Some efficient fourth order multipoint methods for solving equations, *BIT* **9** (1969), 119–124.
- [56] P. Jarratt, A review of methods for solving nonlinear algebraic equations in one variable. In: *Numerical Methods for Nonlinear Algebraic Equations*, Ph. Rabinowitz (ed.), Gordon and Breach, London, New York, Paris, 1970.
- [57] T. A. Jeeves, Secant modifications of Newton's method, *Comm. Assoc. Comput. Mach.* **1**(8) (1958), 9–10.
- [58] L. V. Kantorovich, On Newton's method, *Trudy Mat. Inst. Steklov* **28** (1949), 104–144.
- [59] L. V. Kantorovich, On some further applications of the Newton approximation method (Russian), *Vestnik Leningrad Univ.* **7** (1957), 68–103.
- [60] I. O. Kerner, Algorithm 238, *Comm. Assoc. Comput. Mach.* **9** (1966), 273.
- [61] W. M. Kincaid, Solution of equations by interpolation, *Ann. Math. Statist.* **19** (1948), 207–219.
- [62] R. F. King, A fifth-order family of modified Newton methods, *BIT* **11** (1971), 409–412.
- [63] R. F. King, Tangent methods for nonlinear equations, *Numer. Math.* **18** (1972), 298–304.
- [64] R. F. King, An improved Pegasus method for root finding, *BIT* **13** (1973), 423–427.
- [65] R. F. King, A family of fourth-order methods for nonlinear equations, *SIAM J. Numer. Anal.* **10** (1973), 876–879.
- [66] R. F. King, Methods without secant steps for finding a bracketed root, *Computing* **17** (1976), 49–57.
- [67] I. Kiss, Über eine allgemeinerung des Newtonschen Näherungsfahren, *Z. Angew. Math. Mech.* **34** (1954), 68–69.
- [68] S. Kulik, On the solution of algebraic equations, *Proc. Amer. Math. Soc.* **10** (1959), 185–192.
- [69] H. T. Kung and J. F. Traub, Optimal order of one-point and multipoint iterations, *J. Assoc. Comput. Mach.* **21** (1974), 643–651.
- [70] H. Kung and J. F. Traub, Optimal order and efficiency for iterations with two evaluations, *SIAM J. Numer. Anal.* **13** (1976), 84–99.
- [71] E. N. Laguerre, *Oeuvres de Laguerre*, Vol. 1, pp. 87–103.
- [72] E. N. Laguerre, Sur une methode pour obtenir par approximation les racines d'une equation algebrique qui a toutes ses racines reelles, *Nouvelles Ann. de Math. 2e series* **19** (1880), 88–103.
- [73] D. K. Lika, Iteration methods of high order, Abstracts of papers read at the Second Scientific-Tech. Republ. Conf., Moldavia, Kishinev, 1965, pp. 13–16.
- [74] S. Lin, A method of finding roots of algebraic equations, *J. Math. and Phys.* (now *Studies in Appl. Math.*) **22** (1943), 60–77.
- [75] G. V. Milovanovic and Dj. R. Djordjevic, The solution of non-linear equations using iterative processes derived from exponential approximation (Serbo-Croatian), *Proc. 4th Bos.-Herc. Symp. on Informatics*, Jahorina, March 24–28, 1980, 2, 465.1–465.5.
- [76] D. E. Muller, A method for solving algebraic equations using an automatic computer, *Math. Comp.* **10** (1956), 208–215.

- [77] T. Murakami, Some fifth-order multipoint iterative formulae for solving equations, *J. of Information Processing* **1** (1978), 138–139.
- [78] D. Nerinckx and A. Haegemans, A comparison of non-linear equation solvers, *J. Comp. Appl. Math.* **2** (1976), 145–148.
- [79] P. F. Nesdore, The determination of an algorithm which uses the mixed strategy technique for the solution of single nonlinear equations. In: *Numerical Methods for Nonlinear Algebraic Equations*, Ph. Rabinowitz (ed.), Gordon and Breach, London, 1970.
- [80] B. Neta, A sixth-order family of methods for nonlinear equations, *Intern. J. Computer Math.* **7** (1979), 157–161.
- [81] B. Neta, On a family of multipoint methods for nonlinear equations, *Intern. J. Computer Math.* **9** (1981), 353–361.
- [82] B. Neta, A new family of higher order methods for solving equations, *Intern. J. Computer Math.* **14** (1983), 191–195.
- [83] A. W. M. Nourein, Root determination by use of Padé approximants, *BIT* **16** (1976), 291–297.
- [84] A. M. Ostrowski, On approximation of equations by algebraic equations, *SIAM J. Numer. Anal.* **1** (1964), 104–130.
- [85] A. M. Ostrowski, *Solution of Equations and Systems of Equations*, 3rd ed., Academic Press, New York, London, 1973.
- [86] D. B. Popovski, A hybrid algorithm for finding roots, *Informatica* **3** (1979), 16–17.
- [87] D. B. Popovski, A Fortran IV subroutine for finding a bracketed root, *Informatica* **4** (1979), 23–24.
- [88] D. B. Popovski, A root finding algorithm, *Prilozi* **30–31** (1979), 289–293.
- [89] D. B. Popovski, Solution of equations on the basis of one-point logarithmic approximation, Proc. 14th Yug. Symp. on Information Processing, *Informatica* **79** (October 1–6, 1979), 3/202.
- [90] D. B. Popovski, Numerical solution of equations on the basis of approximation by the curve $(x - p_1)[y(x) - p_2]^2 - p_3 = 0$, *Int. J. Num. Math. Engng.* **14** (1979), 1574.
- [91] D. B. Popovski and P. B. Popovski, On Jarratt's two-point method for finding roots, *Proc. 4th Bos.-Herc. Symp. on Informatics*, Jahorina, March 24–28 **2** (1980), 413.1–413.5.
- [92] D. B. Popovski, Hybrid Chambers-bisection algorithm for finding a bracketed root, *Contributions (Society of Sciences and Art—Bitola, Yugoslavia)* **32–33** (1980), 109–113.
- [93] D. B. Popovski, A family of one-point iteration formulae for finding roots, *Intern. J. Computer Math.* **8** (1980), 85–88.
- [94] D. B. Popovski, An extension of Chebyshev's iteration, *Informatica* **4** (1980), 26–28.
- [95] D. B. Popovski, On an algorithm for finding function zeros (Serbo-Croatian), *Informatica* **1** (1980), 47–48.
- [96] D. B. Popovski, On a subroutine for root finding, *Informatica* **3** (1980), 23–24.
- [97] D. B. Popovski, A method for solving equations, *Informatica* **1** (1981), 212–213.
- [98] D. B. Popovski, Method of parabolic approximation for solving the equation $x = f(x)$, *Intern. J. Computer Math.* **9** (1981), 243–248.

- [99] D. B. Popovski, An improvement of the Ostrowski root finding method, *Z. Angew. Math. Mech.* **61** (1981), T303–T305.
- [100] D. B. Popovski, Method of tangential hyperbolic approximation for solving equations, *Proc. 3rd Int. Symp.—Computers at the University, Cavtat, May 25–28 (1981)*, 311.1–311.6.
- [101] D. B. Popovski, A note on King's fifth order family of methods for solving equations, *BIT* **21** (1981), 129–130.
- [102] D. B. Popovski, A two-step method for solving equations (Macedonian), *Proc. College of Engineering, University of Bitola, Yugoslavia* **1** (1981), 79–83.
- [103] D. B. Popovski, A method for solving equations, *Informatica* **1** (1981), 212–213.
- [104] D. B. Popovski, A note on Neta's family of sixth-order methods for solving equations, *Intern. J. Computer Math.* **10** (1981), 91–93.
- [105] D. B. Popovski, A class of two-step methods for solving equations, *Proc. 4th Int. Symp.—Computer at the University, Cavtat, May 24–28 (1982)*, 487–491.
- [106] D. B. Popovski, Sixth order methods for solving equations, *J. Appl. Math. Phys. (ZAMP)* **33** (1982), 434–438.
- [107] D. B. Popovski and P. B. Popovski, Some new one-point iteration functions of order three for solving equations, *Z. Angew. Math. Mech.* **62** (1982), T344–T345.
- [108] D. B. Popovski, A note on King's method F for finding a bracketed root, *Computing* **29** (1982), 355–359.
- [109] E. J. Putzer, A numerical method for solving scalar equations, *Amer. Math. Monthly* **69** (1962), 408–411.
- [110] A. Renyi, On Newton's method of approximation (Hungarian), *Mat. Lapok* **1** (1950), 278–293.
- [111] H. W. Richmond, On certain formulae for numerical approximation, *J. London Math. Soc.* **19** (1944), 31–38.
- [112] J. Rissanen, On optimum root-finding algorithms, *J. Math. Anal. Appl.* **36** (1971), 220–225.
- [113] G. S. Salehov, On the convergence of the process of tangent hyperbolas (Russian), *Doklady Akad. Nauk SSSR* **82** (1952), 525–528.
- [114] E. Schroder, Über unendlich viele Algorithmen zur Auflösung der Gleichungen, *Math. Ann.* **2** (1870), 317–365.
- [115] B. T. Smith, ZERPOL, a zero finding algorithm for polynomials using Laguerre's method, *Proc. 1967 Army Numerical Analysis Conference*, Madison, WI (1967), 153–174.
- [116] J. N. Snyder, Inverse interpolation, a real root of $f(x)=0$, University of Illinois Digital Computer Laboratory, ILLIAC I Library Routine H1-71 (1953), 4 pp.
- [117] R. W. Snyder, One more correction formula, *Amer. Math. Monthly* **62** (1955), 722–725.
- [118] J. F. Steffensen, Remarks on iteration, *Skand. Aktuar. Tidskr.* **16** (1934), 64.
- [119] J. K. Stewart, Another variation of Newton's method, *Amer. Math. Monthly* **58** (1951), 331–334.
- [120] W. M. Stone, A form of Newton's method with cubic convergence, *Quart. Appl. Math.* **11** (1953), 118–119.

- [121] J. F. Traub, On functional iteration and calculation of roots, 16th National Assoc. Comput. Mach. Conference, 1961.
- [122] J. F. Traub, On a class of iteration formulas and some historical notes, *Comm. Assoc. Comput. Mach.* **4**(6) (1961), 276–278.
- [123] J. F. Traub, The theory of multipoint iteration functions, *Digest of Technical Papers*, Assoc. Comput. Mach. 1962 National Conference, 1962, pp. 80–81.
- [124] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, New York, 1964.
- [125] V. A. Varyukhin and S. A. Kas'yanyuk, Iterative methods of determining the roots of equations more accurately, *USSR Comp. Math. & Math. Phys.* **9** (1969), 247–252.
- [126] V. I. Verbuk and D. I. Milman, Wegstein's method as a modification to the secant method (Russian), *Zhur. Vichisl. Mat. i Mat. Fiz.* **17**(2) (1977), 507; *USSR Comp. Math. Math. Phys.* **17** (1977), 215.
- [127] H. D. Victory and B. Neta, A higher order method for multiple zeros of nonlinear functions, *Intern. J. Computer Math.* **12** (1983), 329–335.
- [128] H. S. Wall, A modification of Newton's method, *Amer. Math. Monthly* **55** (1948), 90–94.
- [129] J. H. Wegstein, Accelerating convergence of iterative processes, *Comm. Assoc. Comput. Mach.* **1**(6) (1958), 9–13.
- [130] J. H. Wegstein, Algorithm 2, *Comm. Assoc. Comput. Mach.* **3**(2) (1960), 74.
- [131] W. Werner, Über ein Verfahren der Ordnung $1+\sqrt{2}$ zur Nullstellenbestimmung, *Numer. Math.* **32** (1979), 333–342.
- [132] W. Werner, Some improvement of classical iterative methods for the solution of nonlinear equations. In: *Proc. Numer. Solution of Nonlinear Equations*, E. L. Allgower, K. Glashoff and N.-O. Petigen (eds.), Bremen 1980, Springer-Verlag, pp. 426–440.
- [133] W. Werner, Iterationsverfahren hoherer Ordnung zur Lösung nichtlinearer Gleichungen, *Z. Angew. Math. Mech.* **61** (1981), T322–T324.
- [134] W. Werner, Some efficient algorithms for the solution of a single nonlinear equation, *Intern. J. Computer Math.* **9** (1981), 141–149.
- [135] W. Werner, Some supplementary results on the $1+\sqrt{2}$ order method for the solution of nonlinear equations, *Numer. Math.* **38** (1982), 383–392.
- [136] W. Werner, On higher order iterative methods for the solution of nonlinear equations, submitted for publication.
- [137] E. T. Whittaker, A formula for the solution of algebraic or transcendental equations, *Proc. Edinburgh Math. Soc.* **36** (1918), 103–106.
- [138] D. Woodhouse, A note on the secant method, *BIT* **15** (1975), 323–327.
- [139] J. H. Wilkinson, Two algorithms based on successive linear interpolation, Technical Report CS60, Computer Science Department, Stanford University, 1967.
- [140] P. Wynn, On a cubically convergent process for determining the zeros of certain functions, *Math. Table & Other Aids Comp.* **10** (1956), 97–100.